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The lower domination parameters in inflation of graphs of radius 1

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Abstract

The inflation G_1 of a graph G is the line graph of the subdivision of G . If G is a complete graph the equality $\text{ir}(G_1) = \gamma(G_1)$ was proved by Favaron in 1998. We conjectured that the equality holds when G is any graph of radius 1. But it turned out that it is not true. Moreover, we proved that for the class of radius 1 graphs there does not exist a better upper bound for the relation $\gamma(G_1)/\text{ir}(G_1)$ than $\frac{3}{2}$. We found also a sufficient condition for the equality $\gamma(G_1) = \text{ir}(G_1)$.
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We consider only simple graphs. Let $G = (V(G), E(G))$ be a graph of order $n(G)$ and diameter $\text{diam}(G)$. The neighbourhood and closed neighbourhood of a vertex x of G are, respectively, denoted by $N(x)$ and $N[x]$ (with $N[x] = N(x) \cup \{x\}$). If $X \subseteq V(G)$, $G[X]$ is the subgraph induced in G by X . Let $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. A set X of vertices of G is *dominating* if $N[X] = V(G)$. The minimum cardinality of a dominating set is denoted by $\gamma(G)$.

If x is a vertex of a subset X of $V(G)$, the set $PN(x) = N[x] \setminus N[X \setminus \{x\}]$ is called the *X-private neighbourhood* of x and its elements are the *X-private neighbours* of x . The vertex x of X is *irredundant* in X if its *X-private neighbourhood* is not empty, *redundant* otherwise. The set X is *irredundant* in G if all its vertices are irredundant. If an irredundant set X is *maximal* for the inclusion, then for any vertex u which is not dominated by X there exists a non-isolated vertex y of X which is redundant in $X \cup \{u\}$. That means that u dominates every *X-private neighbour* of y . We say that u *annihilates* y . The minimum cardinality of a maximal irredundant set of G is denoted

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by $\text{ir}(G)$. Since any minimal dominating set is a maximal irredundant set, we have for any graph G

$$\text{ir}(G) \leq \gamma(G). \quad (1)$$

The *inflation* G_1 of a graph G is the line graph of the subdivision of G . The subdivision of G is the graph obtained from G by replacing each edge by a path of length 2. We denote by X the clique in G_1 corresponding to vertex x in G . If xy is an edge in G then we denote (xy) and (yx) the vertices in X and Y corresponding to this edge. An edge $(xy)(yx)$ will be a blue edge while (xy) will be a blue neighbour of (yx) , the other vertices adjacent to (xy) will be the red neighbours (that are vertices in X).

Dunbar and Haynes [1] proved that for every graph H , $\text{diam}(H) \leq 2$ if and only if $\gamma(H_1) = n(H) - 1$.

If G is a complete graph the equality $\text{ir}(G_1) = \gamma(G_1)$ was proved in [2] by Favaron. She conjectured that the same equality holds in the case when G is a tree, which was proved by Puech [4].

It is interesting to find other classes of graphs for which this equality holds.

Let G be a graph of radius 1 that is $\gamma(G) = 1$. Let c be a vertex which is adjacent to all other vertices of G . Then in G_1 the vertex c is replaced by a clique C that contains $n - 1$ vertices.

We conjectured that the equality holds when G is any graph of radius 1. But it turned out that it is not true. Moreover, we proved that for the class of radius 1 graphs there does not exist a better upper bound for the relation $\gamma(G_1)/\text{ir}(G_1)$ than $\frac{3}{2}$. It was shown in [3] that the bound $\frac{3}{2}$ is the best possible in the class of claw-free graphs and

$$\frac{\gamma(G_1)}{\text{ir}(G_1)} < \frac{3}{2} \quad (2)$$

in the class of inflations.

Theorem 1. *There exists a series $\{G^k\}$ of graphs of radius 1, such that*

$$\lim_{k \rightarrow \infty} \frac{\gamma(G_1^k)}{\text{ir}(G_1^k)} = \frac{3}{2}.$$

Proof. For $k \geq 2$ graph, G^k consists of k paths (a_i, b_i, c_i) for $1 \leq i \leq k$, vertices of different paths are not adjacent, and a vertex c adjacent to all vertices of the paths, mentioned. It is clear from the definition that every G^k is a graph of radius 1. It is obvious that $\text{diam}(H) \leq 2$ for every graph H of radius 1, hence $\gamma(G_1^k) = n(G^k) - 1 = 3k$ by Dunbar and Haynes [1]. Consider the following set $W = \{(cb_1), (b_1a_1), (b_1c_1), \dots, (b_ka_k), (b_kc_k)\}$ in G_1^k , for $k \geq 2$. W is irredundant, for (cb_1) is isolated in W , and the W -private neighbours of the other vertices are their blue neighbours. For an arbitrary vertex $v \in V \setminus W$ there exists one of the three possibilities:

- (1) $v \in C \setminus \{(cb_1)\}$, then v dominates (cb_1) and its private neighbourhood;
- (2) $v \in B_i$, for some i , $1 \leq i \leq k$, in this case $N[v] \subset N[W]$;

- (3) $v \in A_i$ (resp. C_i), for $1 \leq i \leq k$, then either v is a unique W -private neighbour of $(b_i a_i)$ (resp. $(b_i c_i)$) or v dominates this private neighbour.

In all cases $W \cup \{x\}$ is redundant, therefore W is a maximal irredundant set. So $\text{ir}(G_1^k) \leq |W| = 2k + 1$.

Then using (2) we get

$$\frac{3k}{1+2k} \leq \frac{\gamma(G_1^k)}{\text{ir}(G_1^k)} < \frac{3}{2}$$

and

$$\lim_{k \rightarrow \infty} \frac{\gamma(G_1^k)}{\text{ir}(G_1^k)} = \frac{3}{2}. \quad \square$$

Theorem 2. Let G be a graph of radius 1, c be a vertex in G , adjacent to all other vertices of G , and C be the clique in G_1 which replaces the vertex c in G . Let W be a maximal irredundant set in G_1 with $|W| = \text{ir}(G_1)$. Then $|W \cap C| \neq 1$ implies $\gamma(G_1) = \text{ir}(G_1) = n(G) - 1$.

Proof. We use the result of Dunbar and Haynes [1] as in Theorem 1 to get $\gamma(G_1) = n(G) - 1$. An obvious consequence of the last equality and inequality (1) is $\text{ir}(G_1) \leq n(G) - 1$. Also note that $|C| = n(G) - 1$.

Assume first $|W \cap C| = 0$. We want to prove that $|W| \geq n(G) - 1$. If $v \in C$ then it is either a private neighbour of some vertex in W , or a vertex in $V \setminus N[W]$. Let C_1 be a set of vertices with the first property, $C_2 = C \setminus C_1$.

Next, we prove that there exists an injective mapping φ from C into W . For $v \in C_1$ let $\varphi(v)$ be the blue neighbour of v . Consider $v \in C_2$. Let B be the clique containing the blue neighbour of v . The vertex (bc) must annihilate some vertex in W , otherwise $W \cup \{(bc)\}$ is irredundant, because in this case v would be a $(W \cup \{(bc)\})$ -private neighbour for (bc) . Let $\varphi(v)$ be one of these vertices annihilated by (bc) . $PN(\varphi(v)) \subseteq B$, otherwise $\varphi(v)$ is not annihilated by (bc) .

Let $v_1, v_2 \in C_1, v_1 \neq v_2$ then $\varphi(v_1) \neq \varphi(v_2)$, otherwise $w = \varphi(v_1) = \varphi(v_2)$ would have two blue neighbours, that contradicts the simpleness of G . Now let $v_1 \in C_1, v_2 \in C_2$. The vertex v_1 is a W -private neighbour of $\varphi(v_1)$, hence W -private neighbourhood of $\varphi(v_1)$ intersects C , and W -private neighbourhood of $\varphi(v_2)$ as noted above does not intersect C , thus $\varphi(v_1) \neq \varphi(v_2)$. This time let $v_1, v_2 \in C_2, v_1 \neq v_2$. B_1 (resp. B_2) is the clique containing the blue neighbour of v_1 (resp. v_2). If $\varphi(v_1)$ is isolated in W and $\varphi(v_2)$ is not isolated in W , then $\varphi(v_1) \neq \varphi(v_2)$. If $\varphi(v_1)$ and $\varphi(v_2)$ are both isolated or both not isolated in W their W -private neighbourhoods are contained in B_1 and B_2 , respectively, hence are different. Again we get $\varphi(v_1) \neq \varphi(v_2)$. We proved that for arbitrary $v_1, v_2 \in C, v_1 \neq v_2$ implies $\varphi(v_1) \neq \varphi(v_2)$. That is φ is an injective mapping. Hence $|\text{ir}(G_1)| = |W| \geq |\varphi(C)| = n(G) - 1$.

Assume now $|W \cap C| \geq 2$. Let $v \in C \setminus W$. As $|W \cap C| \geq 2$ this vertex is not contained in the W -private neighbourhoods of vertices in $C \setminus W$. Let B be the clique containing the blue neighbour of v . Then B must also contain a vertex from W , otherwise $W \cup \{v\}$ is irredundant.

Again we can define φ as a mapping from C into W :

- (i) for $v \in C \cap W$ we define $\varphi(v) = v$;
- (ii) for $v \in C \setminus W$ we define $\varphi(v)$ as one vertex in $B \cap W$.

So if $v_1, v_2 \in C \setminus W$, with $v_1 \neq v_2$, then $\varphi(v_1)$ and $\varphi(v_2)$ lie in different red cliques. Hence $\varphi(v_1) \neq \varphi(v_2)$. As well as above φ is an injection and we have $\text{ir}(G_1) = |W| \geq |\varphi(C)| = n(G) - 1$. The theorem is proved. \square

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